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On Radius Problems for Analytic Functions of Koebe Type

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Abstract. By virtue of the extremal function $f(z)$ for $S^*(\alpha)$ which is the class of all starlike functions $f(z)$ of order α having $f(0)=0$ and $f'(0)=1$ in the open unit disk U, new function of Koebe type is considered. The object of the present paper is to derive radii for starlikeness of order α, and for convexity of order α for the function of Koebe type. Using the extremal functions for the classes of α-spiral like of order β and of α-convex like of order β, we also consider the analytic function of the generalized Koebe type. Some interesting examples for the theorems are also given with their mapping properties.

1 Introduction

Let A be the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

that are analytic in the open unit disk $U=\{z:z\in\mathbb{C}\text{ and }|z|<1\}$. A function $f(z)$ in A is said to be starlike of order α if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha$$

for some α $(0\leqq\alpha<1)$ and all z in U. We denote by $S^*(\alpha)$ the subclass of A consisting of all starlike functions of order α in U. A function $f(z)$ in A is said to be convex of order α if it satisfies

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha$$

for some α $(0\leqq\alpha<1)$ and all z in U. Also we denote by $K(\alpha)$ the subclass of A consisting of functions $f(z)$ which are convex of order α in U. In particular, we denote by $S^*(0)\equiv S^*$ and $K(0)\equiv K$ (cf. Robertson [3]). By Robertson [3], we note that

(i) $f(z)=\dfrac{z}{(1-z)^{2(1-\alpha)}}$ is the extremal function for the class $S^*(\alpha)$.

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$$(\text{ ii }) \ f(z) = \begin{cases} \dfrac{1-(1-z)^{2\alpha-1}}{2\alpha-1} & (\alpha \neq \dfrac{1}{2}) \\ -\log(1-z) & (\alpha = \dfrac{1}{2}) \end{cases}$$

is the extremal function for the class $K(\alpha)$.

If we take $\alpha = 0$ in (i) and (ii), then we see that

(iii) $f(z) = \dfrac{z}{(1-z)^2}$ is the extremal function for the class S^*.

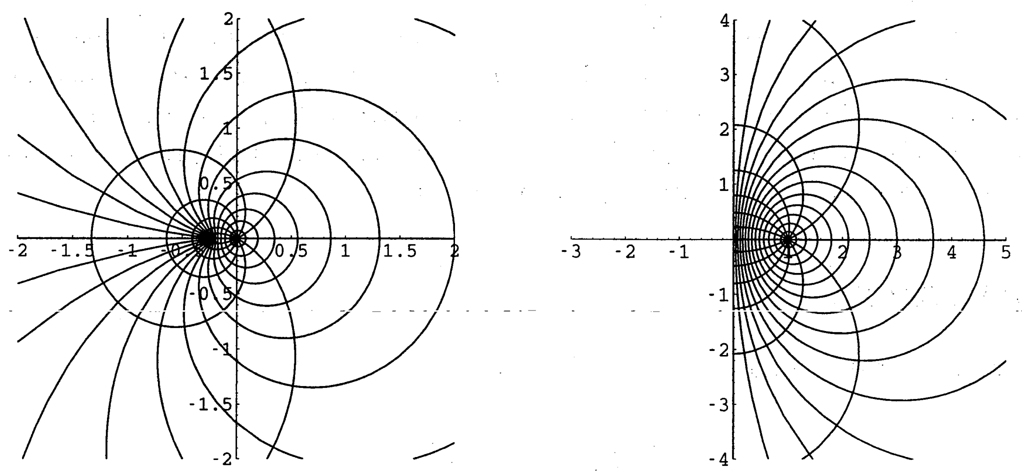

Fig 1.1: Image of $|z| = r$ by $f(z) = \dfrac{z}{(1-z)^2}$ (left), $\dfrac{zf'(z)}{f(z)}$ (right) $\quad$ ($r = 1$ in all cases).

(iv) $f(z) = \dfrac{z}{1-z}$ is the extremal function for the class K.

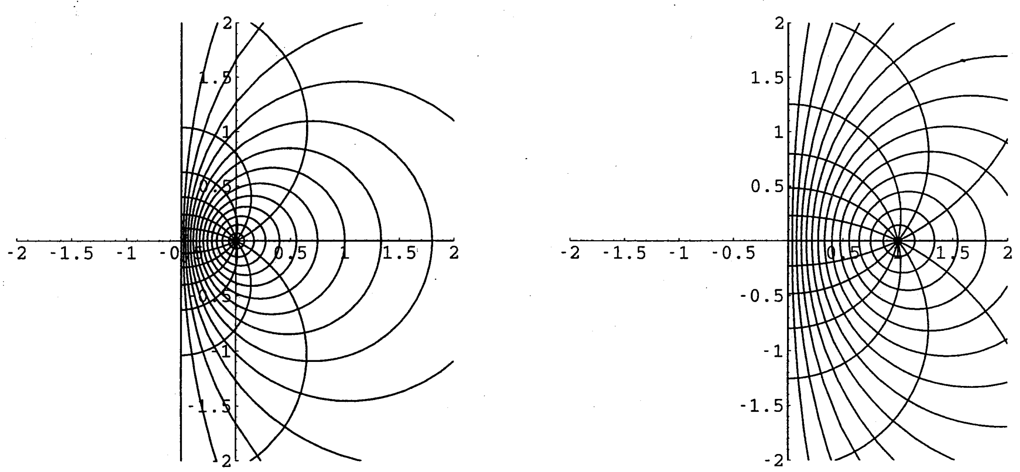

Fig 1.2: Image of $|z| = r$ by $f(z) = \dfrac{z}{1-z}$ (left), $1 + \dfrac{zf''(z)}{f'(z)}$ (right) $\quad$ ($r = 1$ in all cases).

Furthermore, by Marx [2] and Strohhäcker [4] (also by Komatu [1]), we see that K is the subclass of $S^*(\frac{1}{2})$. And by Wilken and Feng [5], $K(\alpha)$ is the subclass of $S^*(\beta(\alpha))$, where

$$\beta(\alpha) = \begin{cases} \dfrac{2\alpha - 1}{2(1 - 2^{1-2\alpha})} & (\alpha \neq \dfrac{1}{2}) \\ \dfrac{1}{2\log 2} & (\alpha = \dfrac{1}{2}). \end{cases}$$

In view of the previous properties for the classes $S^*(\alpha)$ and $K(\alpha)$, it is very interesting to consider the following analytic function

$$f(z) = \frac{z}{(1-z)^k} \qquad (k \in \mathbb{R})$$

which was called Koebe type. Then by the extremal functions $f(z)$ for the classes $S^*(\alpha)$ and $K(\alpha)$, we know that, in general, this function $f(z)$ is not univalent (so, is not starlike or convex) in U. But, since every analytic function $f(z)$ maps, one-to-one, a small disk onto a small disk, we consider the radius problems for the analytic function $f(z)$ of Koebe type to be starlike and convex of order α.

2 Radii for starlikeness of order α

We derive radii of starlikeness of order α for the function $f(z)$ of Koebe type to be in the class of $S^*(\alpha)$. Our first result is contained in

Theorem 1. *The function $f(z)$ of Koebe type satisfies*

(1) $k > 2(1-\alpha) \implies f(z) \in S^*(\alpha)$ *for* $0 \leqq r < \dfrac{1-\alpha}{k-(1-\alpha)}$ $(|z| = r)$,

(2) $0 \leqq k \leqq 2(1-\alpha) \implies f(z) \in S^*(\alpha)$ *for* $0 \leqq r < 1$ $(|z| = r)$,

(3) $k < 0 \implies f(z) \in S^*(\alpha)$ *for* $0 \leqq r < \dfrac{1-\alpha}{1-\alpha-k}$ $(|z| = r)$.

Proof. By a simple calculation, we have

$$\frac{zf'(z)}{f(z)} = 1 + k\,\frac{z}{1-z}.$$

Letting $z = re^{i\theta}$, and we have

$$\begin{aligned} \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) &= \mathrm{Re}\left(1 + k\,\frac{re^{i\theta}}{1-re^{i\theta}}\right) \\ &= 1 - k\,\frac{r^2 - r\cos\theta}{1 + r^2 - 2r\cos\theta}. \end{aligned}$$

We define the function $g(\theta)$ by

$$g(\theta) = \frac{r^2 - r\cos\theta}{1 + r^2 - 2r\cos\theta}.$$

For $k \geqq 0$, we calculate maximum $g(\theta)$ to get the minimum value for $\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)$. Changing $\cos\theta$ by t $(-1 \leqq t \leqq 1)$ in $g(\theta)$, we get

$$g(t) = \frac{r^2 - rt}{1 + r^2 - 2rt},$$

and then

$$g'(t) = \frac{r(r^2 - 1)}{(1 + r^2 - 2rt)^2}.$$

Thus we know that $g(t)$ is monotone decreasing because $g'(t)$ is non-positive for $0 \leqq r < 1$. Therefore, $g(t)$ has maximum value at $t = -1$. It follows from the above that

$$\begin{aligned} \max g(\theta) &= \frac{r^2 + r}{1 + r^2 + 2r} \\ &= \frac{r}{1 + r}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) &= 1 - k\,\frac{r^2 - r\cos\theta}{1 + r^2 - 2r\cos\theta} \\ &\geqq \frac{1 - (k-1)r}{1 + r} > \alpha \end{aligned}$$

for r satisfying the following inequality

$$1 - \alpha > (k - (1 - \alpha))r. \tag{2.1}$$

We see that if $k > 1 - \alpha$, then

$$0 \leqq r < \frac{1 - \alpha}{k - (1 - \alpha)},$$

and if $k > 2(1 - \alpha)$, then

$$\frac{1 - \alpha}{k - (1 - \alpha)} < 1,$$

so, we derive the case (1) in Theorem 1.

If $0 \leqq k < 1 - \alpha$, then the inequality (2.1) is always satisfied for all r $(0 \leqq r < 1)$.

If $1 - \alpha \leqq k \leqq 2(1 - \alpha)$, then we have the next inequality

$$1 < \frac{1 - \alpha}{k - (1 - \alpha)}.$$

This gives us that $f(z) \in S^*(\alpha)$ for $0 \leqq r < 1$. Hence we get the result of case (2) in Theorem 1.

If $k < 0$, letting $k = -j$, we have

$$f(z) = \frac{z}{(1 - z)^{-j}} = z(1 - z)^j \qquad (j > 0).$$

Similary, for $k \geqq 0$, we have to consider

$$\frac{zf'(z)}{f(z)} = 1 - j\,\frac{z}{1-z}.$$

This gives that

$$\begin{aligned}\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) &= \mathrm{Re}\left(1 - j\,\frac{re^{i\theta}}{1-re^{i\theta}}\right)\\ &= 1 + j\,\frac{r^2 - r\cos\theta}{1 + r^2 - 2r\cos\theta} = 1 + jg(\theta),\end{aligned}$$

where

$$g(\theta) = \frac{r^2 - r\cos\theta}{1 + r^2 - 2r\cos\theta}.$$

When $g(\theta)$ has its minimum value, then $\mathrm{Re}(\frac{zf'(z)}{f(z)})$ becomes minimum. It is easy to check that $g(\theta)$ has the minimum value at $\cos\theta = 1$ because it is monotone decreasing. Hence, we have

$$\begin{aligned}\min\, g(\theta) &= \frac{r^2 - r}{1 + r^2 - 2r}\\ &= \frac{r}{1-r}.\end{aligned}$$

It follows that

$$\begin{aligned}\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) &= 1 + j\,\frac{r^2 - r\cos\theta}{1 + r^2 - 2r\cos\theta}\\ &\geqq \frac{1-(j+1)r}{1-r} > \alpha\end{aligned}$$

for r satisfying

$$r < \frac{1-\alpha}{j+1-\alpha} < 1.$$

Noting that $j = -k$, we conclude that

$$0 \leqq r < \frac{1-\alpha}{1-\alpha-k},$$

which proves the cases (3) in Theorem 1.

We give some examples of functions $f(z)$ in $S^*(\alpha)$ for Theorem 1.

Example 1.

(1) $f(z) = \dfrac{z}{(1-z)^{\frac{3}{4}}} \in S^*(\frac{2}{3})$ for $0 \leqq r < \frac{4}{5}$,

(2) $f(z) = \dfrac{z}{(1-z)^{\frac{1}{3}}} \in S^*(\frac{1}{4})$ for $0 \leqq r < 1$,

(3) $f(z) = \dfrac{z}{(1-z)^{-5}} \in S^*(\frac{1}{6})$ for $0 \leqq r < \frac{1}{7}$.

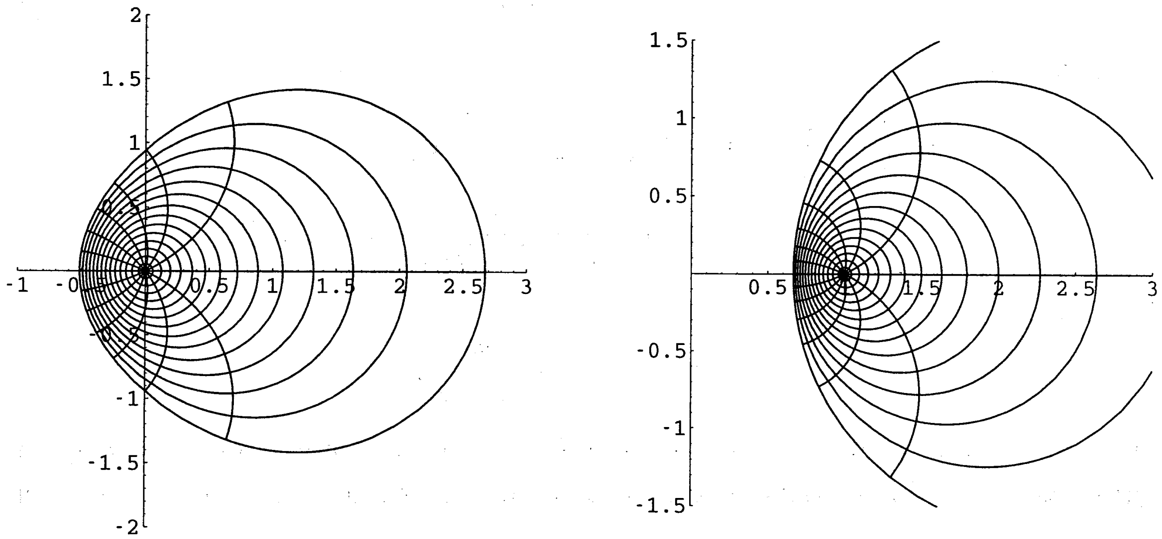

Fig 2.1: Image of $|z| = r$ by $f(z) = \dfrac{z}{(1-z)^{\frac{3}{4}}}$ (left), $\dfrac{zf'(z)}{f(z)}$ (right) $(r = \frac{4}{5}$ in all cases).

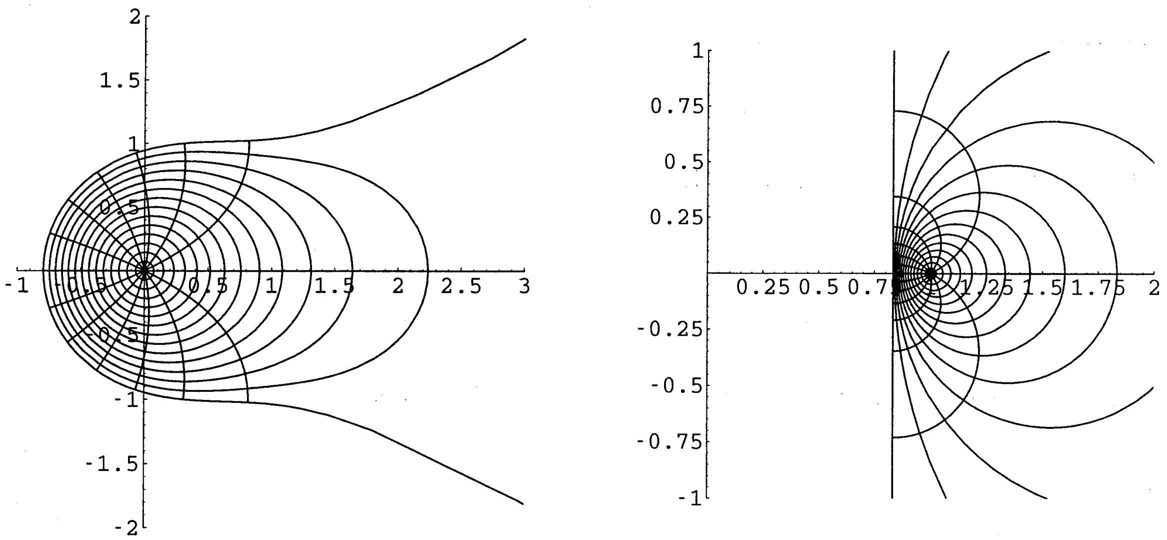

Fig 2.2: Image of $|z| = r$ by $f(z) = \dfrac{z}{(1-z)^{\frac{1}{3}}}$ (left), $\dfrac{zf'(z)}{f(z)}$ (right) $(r = 1$ in all cases).

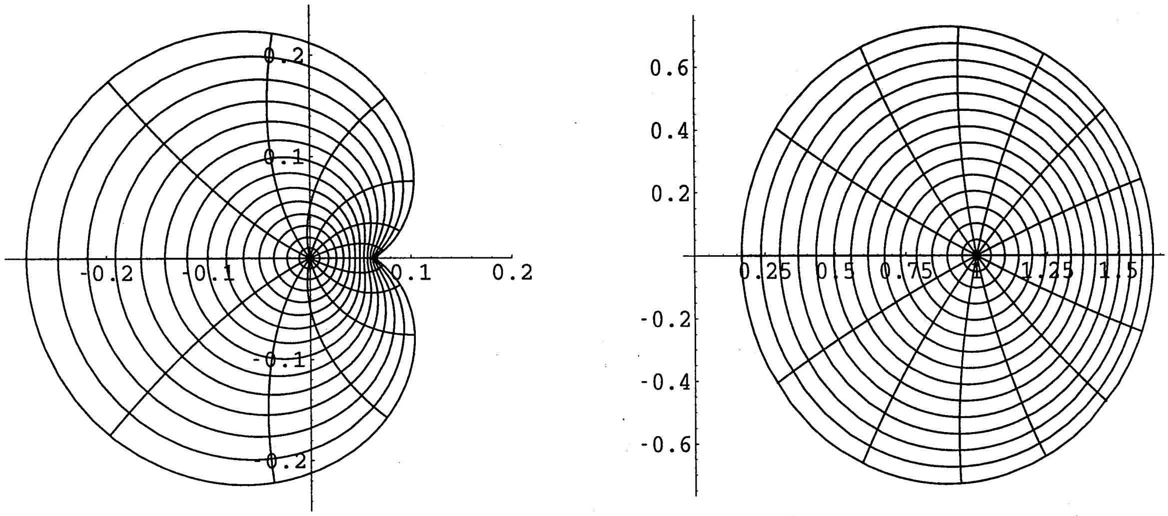

Fig 2.3: Image of $|z| = r$ by $f(z) = \dfrac{z}{(1-z)^{-5}}$ (left), $\dfrac{zf'(z)}{f(z)}$ (right) $(r = \frac{1}{7}$ in all cases).

3 Radii for convexity of order α

Next we discuss the radii of convexity of order α for the function $f(z)$ of Koebe type.

Theorem 2. *The function $f(z)$ of Koebe type satisfies*

(1) $k \geqq 1 \Rightarrow f(z) \in K(\alpha)$ *for*

$$0 \leqq r < \frac{(3-\alpha)k - 2(1-\alpha) - \sqrt{k((\alpha^2 - 2\alpha + 5)k - 4(1-\alpha))}}{2(k-1)(k-(1-\alpha))} \qquad (|z| = r),$$

(2) $k \leqq -1 \Rightarrow f(z) \in K(\alpha)$ *for*

$$0 \leqq r < \frac{2(1-\alpha) - (3-\alpha)k - \sqrt{k((\alpha^2 - 2\alpha + 5)k - 4(1-\alpha))}}{2(1-k)(1-\alpha-k)} \qquad (|z| = r).$$

Proof. From Theorem 1, since

$$\frac{zf'(z)}{f(z)} = \frac{1+(k-1)z}{1-z},$$

we have

$$1 + \frac{zf''(z)}{f'(z)} = (k-1)\,\frac{z}{1+(k-1)z} + \frac{1+kz}{1-z}.$$

Let

$$g_1(z) = \frac{z}{1+(k-1)z} \quad \text{and} \quad g_2(z) = \frac{1+kz}{1-z}.$$

Taking $z = re^{i\theta}$, we see that

$$\begin{aligned}\mathrm{Re}g_1(z) &= \mathrm{Re}\left(\frac{re^{i\theta}}{1+(k-1)re^{i\theta}}\right)\\ &= \frac{r^2(k-1)+r\cos\theta}{1+r^2(k-1)^2+2r(k-1)\cos\theta}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Re}g_2(z) &= \mathrm{Re}\left(\frac{1+kre^{i\theta}}{1-re^{i\theta}}\right)\\ &= \frac{1-r^2k+r(k-1)\cos\theta}{1+r^2-2r\cos\theta}.\end{aligned}$$

Let $h_1(\theta) = \mathrm{Re}g_1(z)$ and $h_2(\theta) = \mathrm{Re}g_2(z)$. Hence we get

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) = (k-1)h_1(\theta)+h_2(\theta).$$

If $k \geqq 1$, when $h_1(\theta)$ and $h_2(\theta)$ take the minimum values for the same θ, $\mathrm{Re}(1+\frac{zf''(z)}{f'(z)})$ has its minimum value. After the calculations, we see

$$\min h_1(\theta) = \frac{-r}{1-r(k-1)} \qquad (\cos\theta = -1 \text{ and } r \leqq \frac{1}{k-1})$$

and

$$\min h_1(\theta) = \frac{r}{1+r(k-1)} \qquad (\cos\theta = 1 \text{ and } r > \frac{1}{k-1}).$$

Similary,

$$\min h_2(\theta) = \frac{1-kr}{1+r} \qquad (\cos\theta = -1).$$

It follows that

$$\begin{aligned}\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) &\geqq (k-1)\frac{-r}{1-r(k-1)}+\frac{1-kr}{1+r}\\ &= \frac{(k-1)^2r^2-(3k-2)r+1}{(1-(k-1)r)(1+r)} > \alpha,\end{aligned}$$

where $r \leqq \dfrac{1}{k-1}$.

Hence, we derive the next inequality

$$(k-1)(k-(1-\alpha))r^2-((3-\alpha)k-2(1-\alpha))r+1-\alpha > 0. \tag{3.1}$$

From (3.1) and $r \geqq 0$,

$$0 \leqq r < \frac{(3-\alpha)k-2(1-\alpha)-\sqrt{k((\alpha^2-2\alpha+5)k-4(1-\alpha))}}{2(k-1)(k-(1-\alpha))},$$

which completes the case (1) of Theorem 2.

If $0 \leqq k < 1$, when $h_1(\theta)$ is maximum and $h_2(\theta)$ is minimum for the same θ, $\mathrm{Re}(1+\frac{zf''(z)}{f'(z)})$ becomes minimum. Note that $h_2(\theta)$ is the same as the case (1). For $h_1(\theta)$,

$$\max h_1(\theta) = \frac{r}{1+r(k-1)} \qquad (\cos\theta = 1 \text{ and } r \leqq \frac{1}{1-k})$$

or

$$\max h_1(\theta) = \frac{-r}{1-r(k-1)} \qquad (\cos\theta = -1 \text{ and } r > \frac{1}{1-k}).$$

But $f(z)$ is not univalent in this case for $|z| = r$ does not include the origin. Therefore, k does not exist such that this condition is satisfied in this case.

If $k < 0$, letting $k = -j$ in $f(z)$. Similary to $k \geqq 0$, we have

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1-jz}{1-z} - (j+1)\,\frac{z}{1-(j+1)z}.$$

Let

$$g_3(z) = \frac{1-jz}{1-z} \quad \text{and} \quad g_4(z) = \frac{z}{1-(j+1)z}.$$

By a simple calculation,

$$\begin{aligned} \mathrm{Re}g_3(z) &= \mathrm{Re}\left(\frac{1-jre^{i\theta}}{1-re^{i\theta}}\right) \\ &= \frac{1+r^2j-(j+1)r\cos\theta}{1+r^2-2r\cos\theta} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Re}g_4(z) &= \mathrm{Re}\left(\frac{re^{i\theta}}{1-(j+1)re^{i\theta}}\right) \\ &= \frac{r\cos\theta - r^2(j+1)}{1+r^2(j+1)^2-2r(j+1)\cos\theta} \end{aligned}$$

for $z = re^{i\theta}$. And let $h_3(\theta) = \mathrm{Re}g_3(z)$ and $h_4(\theta) = -\mathrm{Re}g_4(z)$. Then we have

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) = h_3(\theta) + (j+1)h_4(\theta).$$

When $h_3(\theta)$ and $h_4(\theta)$ have the minimum values for the same θ, we see that $\mathrm{Re}(1+\frac{zf''(z)}{f'(z)})$ has its minimum value. After calculations, we know that

$$\min h_3(\theta) = \frac{1+jr}{1+r} \qquad (\cos\theta = -1 \text{ and } 0 < j < 1)$$

or

$$\min h_3(\theta) = \frac{1-jr}{1-r} \qquad (\cos\theta = 1 \text{ and } j \geqq 1).$$

Similary, for $h_4(\theta)$,

$$\min\, h_4(\theta) = \frac{r}{(j+1)r+1} \qquad (\cos\theta = -1 \ \text{ and } \ \frac{1}{j+1} < r)$$

or

$$\min\, h_4(\theta) = \frac{r}{(j+1)r-1} \qquad (\cos\theta = 1 \ \text{ and } \ \frac{1}{j+1} \geqq r).$$

Thus we have to consider two cases for $\cos\theta = -1$ and $\cos\theta = 1$.

If $\cos\theta = -1$, the domain of $f(z)$ is not the simply connected domain because $|z| = r$ does not include the origin. Therefore, $f(z)$ is not univalent. This case is impossible.

If $\cos\theta = 1$, the domain of $f(z)$ is the simply connected domain because $|z| = r$ includes the origin. Hence we have

$$\begin{aligned}
\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) &\geqq \frac{1-jr}{1-r} - (j+1)\,\frac{r}{1-(j+1)r} \\
&= \frac{(j+1)^2r^2 - (3j+2)r + 1}{(1-r)(1-(j+1)r)} > \alpha \qquad (j \geqq 1 \text{ and } 0 \leqq r \leqq \frac{1}{j+1}).
\end{aligned}$$

In view of the above, we have

$$(j+1)(j+1-\alpha)r^2 - ((3-\alpha)j + 2(1-\alpha))r + 1 - \alpha > 0. \tag{3.2}$$

Solving (3.2) for $r \geqq 0$, we obtain that

$$0 \leqq r < \frac{(3-\alpha)j + 2(1-\alpha) - \sqrt{j((\alpha^2 - 2\alpha + 5)j - 4(1-\alpha))}}{2(j+1)(j+1-\alpha)}.$$

Since $j = -k$, this inequality becomes that

$$0 \leqq r < \frac{2(1-\alpha) - (3-\alpha)k - \sqrt{k((\alpha^2 - 2\alpha + 5)k - 4(1-\alpha))}}{2(1-k)(1-\alpha-k)} \qquad (k \leqq -1),$$

which gives the case (2) in Theorem 2. Therefore we complete the proof of the theorem.

We give two examples for Theorem 2 as follows.

Example 2.

(1) $f(z) = \dfrac{z}{(1-z)^{10}} \in K(\frac{1}{3})$ for $0 \leqq r < \frac{19-\sqrt{235}}{126} = 0.00291293\cdots$,

(2) $f(z) = \dfrac{z}{(1-z)^{-4}} \in K(\frac{1}{7})$ for $0 \leqq r < \frac{23-\sqrt{274}}{85} = 0.0758477\cdots$.

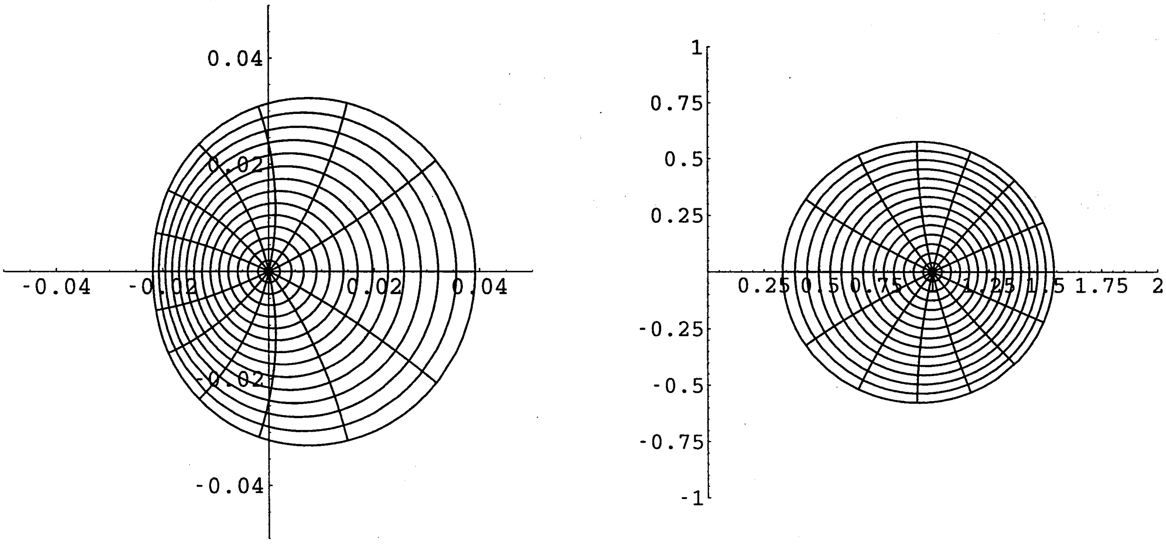

Fig 3.1: Image of $|z| = r$ by $f(z) = \dfrac{z}{(1-z)^{10}}$ (left), $1 + \dfrac{zf''(z)}{f'(z)}$ (right)
(in all cases, $r = \frac{19-\sqrt{235}}{126} = 0.00291293\cdots$).

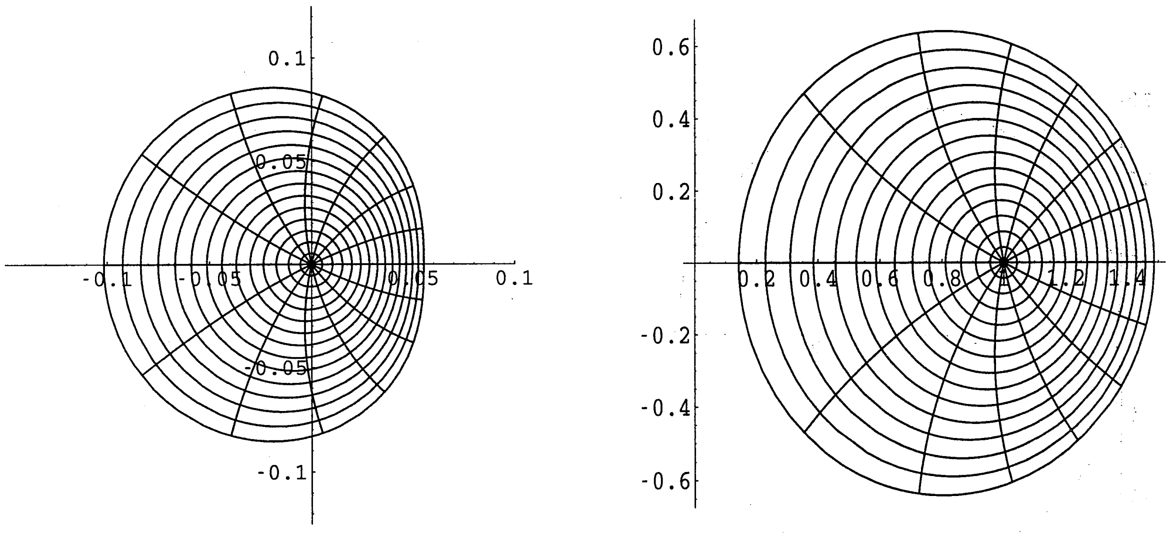

Fig 3.2: Image of $|z| = r$ by $f(z) = \dfrac{z}{(1-z)^{-4}}$ (left), $1 + \dfrac{zf''(z)}{f'(z)}$ (right)
(in all cases, $r = \frac{23-\sqrt{274}}{85} = 0.0758477\cdots$).

By the way, for $-1 < k < 1$ in Theorem 2, we could not specify the bound for the radius r. But we know that every analytic function $f(z)$ in U has the radius r for convexity. For example, the function $f(z)$ given by

$$f(z) = \frac{z}{(1-z)^{\frac{1}{2}}},$$

which is the case $k = \frac{1}{2}$, belongs to K for $0 \leqq r \leqq 0.95$ as follows.

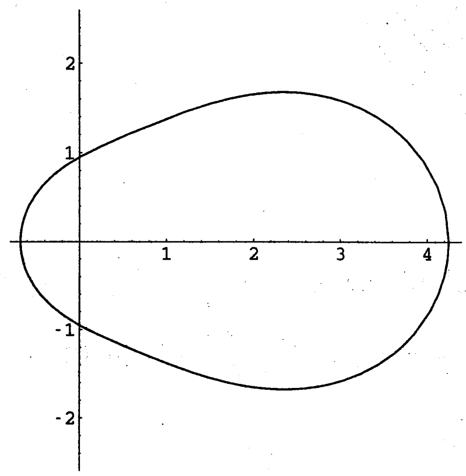

Fig 3.3: Image of $|z| = r$ by $f(z) = \dfrac{z}{(1-z)^{\frac{1}{2}}}$ $(r = 0.95)$.

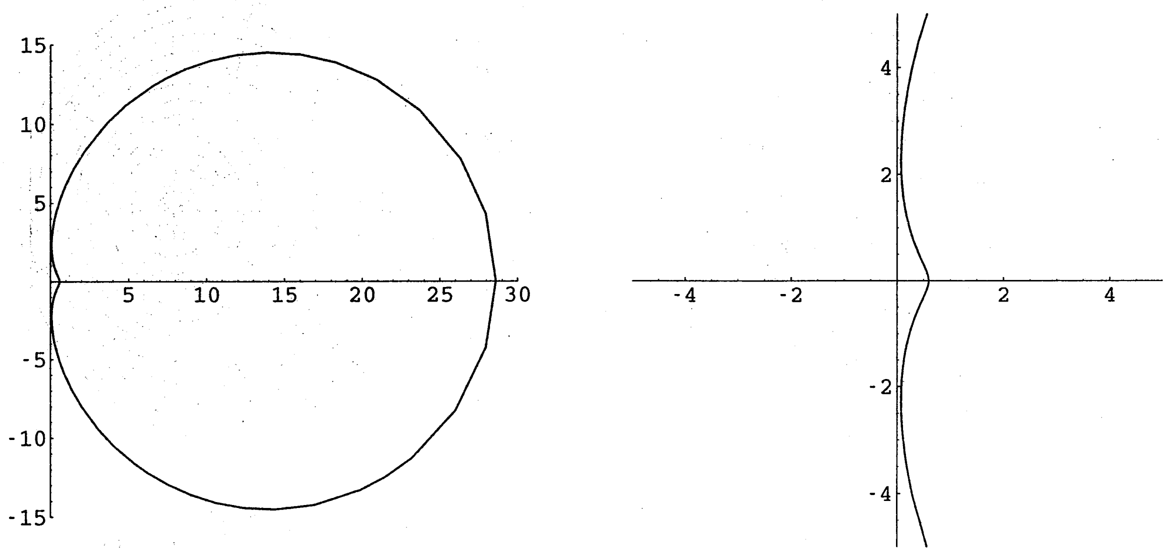

Fig 3.4: Image of $|z| = r$ by $1 + \dfrac{zf''(z)}{f'(z)}$ $(r = 0.95)$.

Thus we give the following problem for convexity of the function $f(z)$ for $-1 < k < 1$.

Problem 1. *Find the sharp bound r for the function $f(z)$ of Koebe type to be convex in $|z| < r$.*

4 Definitions of $S^*_\alpha(\beta)$ and $K_\alpha(\beta)$

A function $f(z) \in A$ is said to be α-spiral like of order β if it satisfies

$$\mathrm{Re}\left(e^{-i\alpha}\frac{zf'(z)}{f(z)}\right) > \beta$$

for some α $(-\frac{\pi}{2} < \alpha < \frac{\pi}{2})$, β $(0 \leqq \beta < \cos\alpha \leqq 1)$ and all z in U. We denote by $S^*_\alpha(\beta)$ the subclass of A consisting of functions $f(z)$ which are α-spiral like of order β in U. A function $f(z)$ in A is said to be α-convex like of order β if it satisfies

$$\mathrm{Re}\left(e^{-i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)\right) > \beta$$

for some α $(-\frac{\pi}{2} < \alpha < \frac{\pi}{2})$, β $(0 \leqq \beta < \cos\alpha \leqq 1)$ and all z in U. Also we denote by $K_\alpha(\beta)$ the subclass of A consisting of funtions $f(z)$ which are α-convex like of order β in U. In particular, we denote by $S^*_0(0) \equiv S^*(0) \equiv S^*$ and $K_0(0) \equiv K(0) \equiv K$.

We can check that the function

$$f(z) = \frac{z}{(1-z)^{2e^{i\alpha}(\cos\alpha-\beta)}} \tag{4.1}$$

is the extremal function for the class $S^*_\alpha(\beta)$. Because, since

$$\frac{zf'(z)}{f(z)} = 1 + 2e^{i\alpha}(\cos\alpha - \beta)\frac{z}{1-z}$$

for the extremal function, we have

$$e^{-i\alpha}\frac{zf'(z)}{f(z)} = e^{-i\alpha} + 2(\cos\alpha - \beta)\frac{z}{1-z}.$$

Note that $w = \frac{z}{1-z}$ maps the unit disk U onto the half domain with $\mathrm{Re}(w) > -\frac{1}{2}$. Therefore, we see that

$$\mathrm{Re}\left(e^{-i\alpha}\frac{zf'(z)}{f(z)}\right) > \cos\alpha - (\cos\alpha - \beta) = \beta.$$

By definitions for the classes $S^*_\alpha(\beta)$ and $K_\alpha(\beta)$, since $f(z) \in K_\alpha(\beta)$ if and only if $zf'(z) \in S^*_\alpha(\beta)$, we calculate the extremal function $f(z)$ for the class $K_\alpha(\beta)$ given by

$$f(z) = \frac{1}{2e^{i\alpha}(\cos\alpha-\beta)-1}\left(\frac{1}{(1-z)^{2e^{i\alpha}(\cos\alpha-\beta)-1}} - 1\right). \tag{4.2}$$

Let us give some examples of functions $f(z)$ in $S^*_\alpha(\beta)$ and $K_\alpha(\beta)$.

Example 3.

(1) $f(z) = \dfrac{z}{(1-z)^{\frac{1+\sqrt{3}i}{2}}}$ (when $\alpha = \dfrac{\pi}{3}$, $\beta = 0$ in (4.1)),

(2) $f(z) = \dfrac{2-\sqrt{3}-(2\sqrt{3}-1)i}{5-2\sqrt{3}}\left(\dfrac{1}{(1-z)^{\frac{2-\sqrt{3}+(2\sqrt{3}-1)i}{4}}} - 1\right)$ (when $\alpha = \dfrac{\pi}{6}$, $\beta = \dfrac{1}{4}$ in (4.2)).

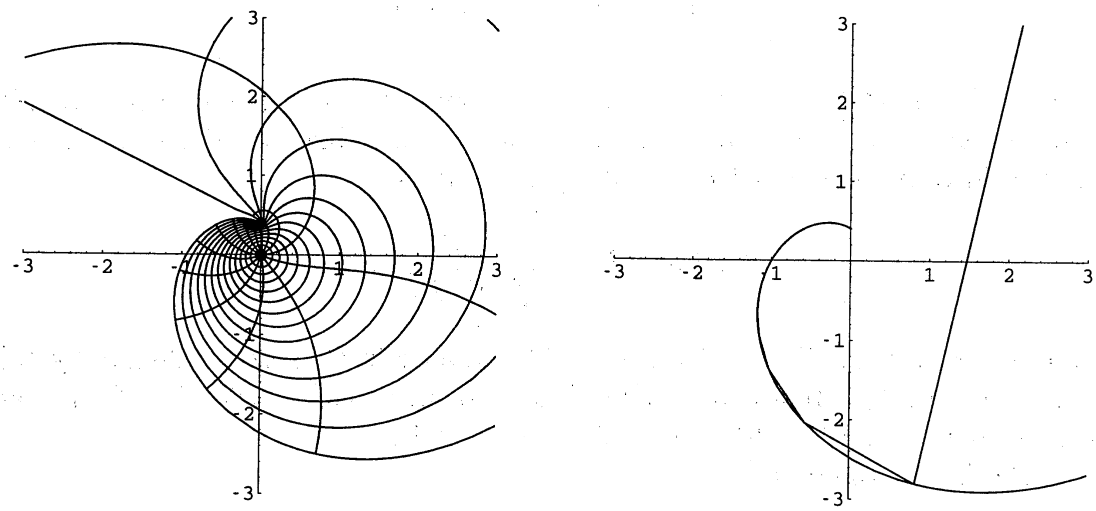

Fig 4.1: Image of $|z| = r$ by $f(z) = \dfrac{z}{(1-z)^{\frac{1+\sqrt{3}i}{2}}}$ ($0 \leqq r \leqq 0.95$ (left), $r = 1$ (right)).

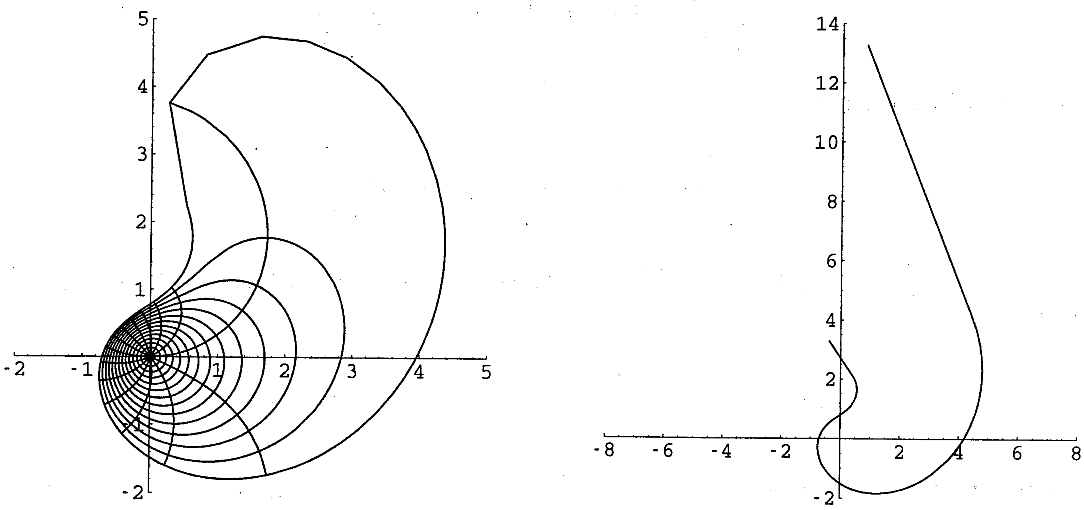

Fig 4.2: Image of $|z| = r$ by $f(z) = \dfrac{2-\sqrt{3}-(2\sqrt{3}-1)i}{5-2\sqrt{3}}\left(\dfrac{1}{(1-z)^{\frac{2-\sqrt{3}+(2\sqrt{3}-1)i}{4}}} - 1\right)$
($0 \leqq r \leqq 0.99$ (left), $r = 1$ (right)).

As we give extremal functions for the classes $S_\alpha^*(\beta)$ and $K_\alpha(\beta)$, our functions in Example 3 show that it is interesting for us to introduce the analytic function of generalized Koebe type by

$$f(z) = \frac{z}{(1-z)^{ke^{i\alpha}}}$$

for some $k \in \mathbb{R}$ and α $(0 \leqq \alpha < 2\pi)$.

If $k = 2(\cos\alpha - \beta)$, then $f(z)$ becomes the extremal function of the class $S_\alpha^*(\beta)$.

5 Radii for α-spiral likeness of order β

We discuss the radii of α-spiral like of order β for the function $f(z)$ of the generalized Koebe type.

Theorem 3. *The function $f(z)$ of the generalized Koebe type satisfies*

$$(1)\ \ k > 2(\cos\alpha - \beta) \Longrightarrow f(z) \in S_\alpha^*(\beta) \quad for \quad 0 \leqq r < \frac{\cos\alpha - \beta}{k - (\cos\alpha - \beta)} \quad (|z| = r),$$

$$(2)\ \ 0 \leqq k \leqq 2(\cos\alpha - \beta) \Longrightarrow f(z) \in S_\alpha^*(\beta) \quad for \quad 0 \leqq r < 1 \quad (|z| = r),$$

$$(3)\ \ k < 0 \Longrightarrow f(z) \in S_\alpha^*(\beta) \quad for \quad 0 \leqq r < \frac{\cos\alpha - \beta}{\cos\alpha - \beta - k} \quad (|z| = r).$$

Proof. By a simple calculation, we have

$$\frac{zf'(z)}{f(z)} = 1 + k\,\frac{e^{i\alpha}z}{1-z},$$

which gives

$$e^{-i\alpha}\,\frac{zf'(z)}{f(z)} = e^{-i\alpha} + k\,\frac{z}{1-z}.$$

Letting $w = \dfrac{z}{1-z}$, we have $z = \dfrac{w}{w+1}$. Since $|z|^2 \leqq |r|^2$ for $|z| \leqq |r|$, we have

$$|z|^2 = \left|\frac{w}{w+1}\right|^2 \leqq |r|^2.$$

After a simple calculation, we have

$$|w|^2 \leqq |w+1|^2\, r^2,$$

which implies

$$\left|w - \frac{r^2}{1-r^2}\right|^2 \leqq \frac{r^2}{(1-r^2)^2}.$$

Hence we have

$$\left| w - \frac{r^2}{1-r^2} \right| \leqq \frac{r}{1-r^2}. \tag{5.1}$$

Now, we can calculate the maximum and minimum values of $\mathrm{Re}(w)$ from (5.1) as follows:

$$\begin{aligned} \max \mathrm{Re}(w) &= \frac{r^2}{1-r^2} + \frac{r}{1-r^2} \\ &= \frac{r}{1-r} \end{aligned}$$

and

$$\begin{aligned} \min \mathrm{Re}(w) &= \frac{r^2}{1-r^2} - \frac{r}{1-r^2} \\ &= -\frac{r}{1+r}. \end{aligned}$$

For $k \geqq 0$, to get the minimum value of $\mathrm{Re}(e^{-i\alpha}\frac{zf'(z)}{f(z)})$, we take the minimum value of $\mathrm{Re}(w)$. Then we see that

$$\begin{aligned} \mathrm{Re}\left(e^{-i\alpha}\frac{zf'(z)}{f(z)}\right) &= \cos\alpha + k\ \mathrm{Re}(w) \\ &\geqq \cos\alpha - k\ \frac{r}{1+r} > \beta. \end{aligned}$$

Since

$$(\cos\alpha - \beta)(1+r) - kr > 0,$$

or

$$(\cos\alpha - \beta - k)r + \cos\alpha - \beta > 0,$$

r satisfies the following inequality

$$(k - (\cos\alpha - \beta))r - (\cos\alpha - \beta) < 0. \tag{5.2}$$

We see that if $k > \cos\alpha - \beta$, then

$$r < \frac{\cos\alpha - \beta}{k - (\cos\alpha - \beta)}$$

and if $k > 2(\cos\alpha - \beta)$, then

$$\frac{\cos\alpha - \beta}{k - (\cos\alpha - \beta)} < 1.$$

So, we derive the case (1) in Theorem 3.

If $0 \leqq k < \cos\alpha - \beta$, then the inequality (5.2) is always satisfied for all r $(0 \leqq r < 1)$.

If $\cos\alpha \leqq k \leqq 2(\cos\alpha - \beta)$, then we have the next inequality

$$1 < \frac{\cos\alpha - \beta}{k - (\cos\alpha - \beta)}.$$

This gives us that $f(z) \in S^*_\alpha(\beta)$ for $0 \leqq r < 1$. Hence we get the result of the case (2) in Theorem 3.

For $k < 0$, to get the minimum value of $\mathrm{Re}(e^{-i\alpha}\frac{zf'(z)}{f(z)})$, we take the maximum value of $\mathrm{Re}(w)$. In this case, we have

$$\begin{aligned}\mathrm{Re}\left(e^{-i\alpha}\frac{zf'(z)}{f(z)}\right) &= \cos\alpha + k\,\mathrm{Re}(w)\\ &\geqq \cos\alpha + k\,\frac{r}{1-r} > \beta.\end{aligned}$$

Similary, for $k \geqq 0$, since

$$(\cos\alpha - \beta)(1-r) + kr > 0,$$

or

$$(k - (\cos\alpha - \beta))r + \cos\alpha - \beta > 0,$$

we have

$$r < \frac{\cos\alpha - \beta}{\cos\alpha - \beta - k}$$

for r satisfying

$$r < \frac{\cos\alpha - \beta}{\cos\alpha - \beta - k} < 1.$$

Thus we have

$$0 \leqq r < \frac{\cos\alpha - \beta}{\cos\alpha - \beta - k},$$

which gives the result of the case (3) in Theorem 3. The proof of Theorem 3 is completed.

We give some examples of functions $f(z)$ in $S^*_\alpha(\beta)$ for Theorem 3.

Example 4.

(1) $f(z) = \dfrac{z}{(1-z)^{3e^{i\frac{\pi}{3}}}} \in S^*_{\frac{\pi}{3}}(\frac{1}{3})$ for $0 \leqq r < \frac{1}{17} = 0.0588235\cdots$,

(2) $f(z) = \dfrac{z}{(1-z)^{\frac{1}{2}e^{i\frac{\pi}{4}}}} \in S^*_{\frac{\pi}{4}}(\frac{1}{5})$ for $0 \leqq r < 1$,

(3) $f(z) = \dfrac{z}{(1-z)^{-7e^{i\frac{\pi}{6}}}} \in S^*_{\frac{\pi}{6}}(\frac{2}{5})$ for $0 \leqq r < \frac{5\sqrt{3}-4}{5\sqrt{3}+66} = 0.0624195\cdots$.

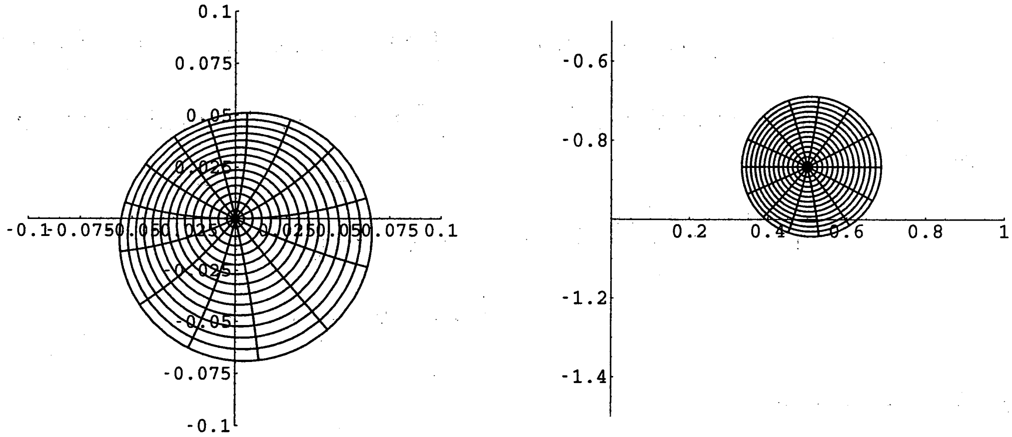

Fig 5.1: Image of $|z| = r$ by $f(z) = \dfrac{z}{(1-z)^{3e^{i\frac{\pi}{3}}}}$ (left), $e^{-i\frac{\pi}{3}}\dfrac{zf'(z)}{f(z)}$ (right) (in all cases, $r = \frac{1}{17} = 0.0588235\cdots$).

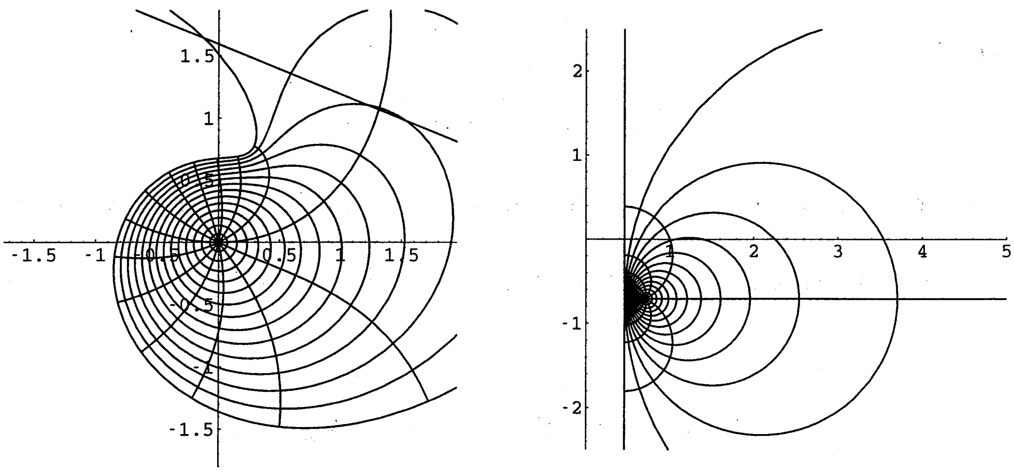

Fig 5.2: Image of $|z| = r$ by $f(z) = \dfrac{z}{(1-z)^{\frac{1}{2}e^{i\frac{\pi}{4}}}}$ (left), $e^{-i\frac{\pi}{4}}\dfrac{zf'(z)}{f(z)}$ (right) (in all cases, $r = 1$).

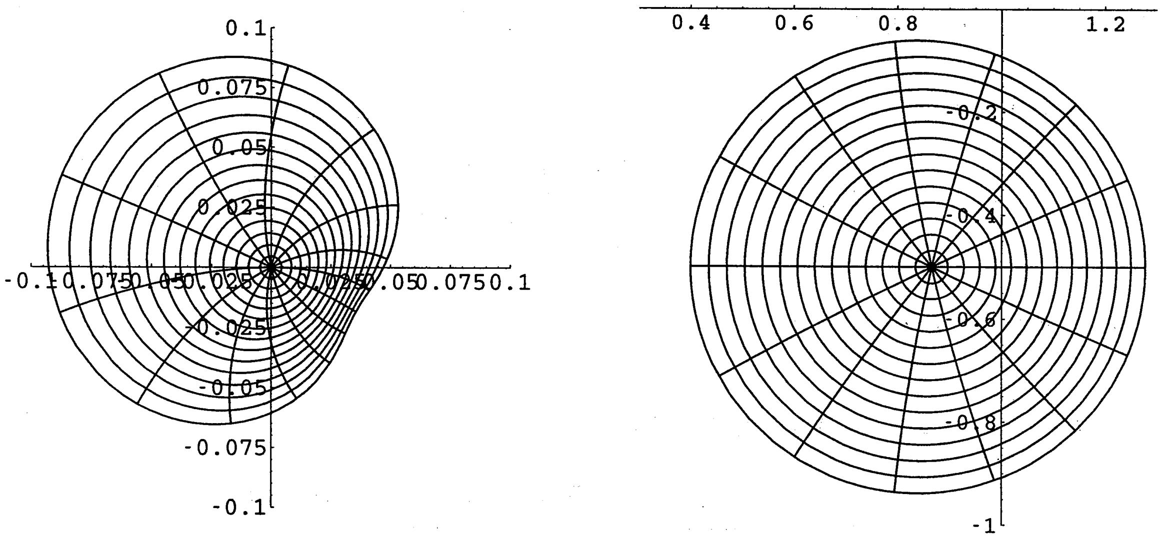

Fig 5.3: Image of $|z| = r$ by $f(z) = \dfrac{z}{(1-z)^{-7e^{i\frac{\pi}{6}}}}$ (left), $e^{-i\frac{\pi}{6}}\dfrac{zf'(z)}{f(z)}$ (right)
(in all cases, $r = \frac{5\sqrt{3}-4}{5\sqrt{3}+66} = 0.0624195\cdots$).

6 Radii for starlikeness of order β

Next we discuss the radii of starlikeness of order β for the function $f(z)$ of the generalized Koebe type.

Theorem 4. *The function $f(z)$ of the generalized Koebe type satisfies*

(1) *for* $k \geqq 0$ *and* $\cos\alpha \geqq 0$,

(i) $k \neq 0$, $\dfrac{1-\beta}{\cos\alpha} \Longrightarrow f(z) \in S^*(\beta)$ *for*

$$0 \leqq r < \frac{k - \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)} \qquad (|z| = r),$$

(ii) $k = \dfrac{1-\beta}{\cos\alpha} \Longrightarrow f(z) \in S^*(\beta)$ *for* $0 \leqq r < \cos\alpha \quad (|z| = r)$,

(iii) $k = 0 \Longrightarrow f(z) \in S^*(\beta)$ *for* $0 \leqq r < 1 \quad (|z| = r)$,

(2) *for* $k \geqq 0$ *and* $\cos\alpha < 0$,

(i) $k \neq 0 \Longrightarrow f(z) \in S^*(\beta)$ *for*

$$0 \leqq r < \frac{k - \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)} \qquad (|z| = r),$$

(3) *for* $k<0$ *and* $\cos\alpha \geqq 0$,

(i) $k<0 \Longrightarrow f(z) \in S^*(\beta)$ *for*

$$0 \leqq r < \frac{k+\sqrt{k^2-4k(1-\beta)\cos\alpha+4(1-\beta)^2}}{2(1-\beta-k\cos\alpha)} \quad (|z|=r),$$

(4) *for* $k<0$ *and* $\cos\alpha < 0$,

(i) $k \neq \dfrac{1-\beta}{\cos\alpha} \Longrightarrow f(z) \in S^*(\beta)$ *for*

$$0 \leqq r < \frac{k+\sqrt{k^2-4k(1-\beta)\cos\alpha+4(1-\beta)^2}}{2(1-\beta-k\cos\alpha)} \quad (|z|=r),$$

(ii) $k = \dfrac{1-\beta}{\cos\alpha} \Longrightarrow f(z) \in S^*(\beta)$ *for* $0 \leqq r < -\cos\alpha \quad (|z|=r)$.

Proof. From Theorem 3, we have

$$\frac{zf'(z)}{f(z)} = 1 + k\,\frac{e^{i\alpha}z}{1-z},$$

Letting $w = \dfrac{e^{i\alpha}z}{1-z}$, that is, $z = \dfrac{w}{w+e^{i\alpha}}$, we have

$$|z|^2 = \left|\frac{w}{w+e^{i\alpha}}\right|^2 \leqq |r|^2 \qquad (|z| \leqq r).$$

After calculations, we have

$$|w|^2 \leqq |w+e^{i\alpha}|^2\, r^2,$$

that is

$$\left|w - \frac{r^2}{1-r^2}\,e^{i\alpha}\right|^2 \leqq \frac{r^2}{(1-r^2)^2}.$$

Hence, we have

$$\left|w - \frac{r^2}{1-r^2}e^{i\alpha}\right| \leqq \frac{r}{1-r^2}. \tag{6.1}$$

Now, we have to calculate the maximum and minimum values of $\mathrm{Re}(w)$ from (6.1). Note that

$$\begin{aligned}\max \mathrm{Re}(w) &= \mathrm{Re}\left(\frac{r^2}{1-r^2}e^{i\alpha}\right) + \frac{r}{1-r^2}\\ &= \frac{r^2\cos\alpha}{1-r^2} + \frac{r}{1-r^2} = \frac{r(r\cos\alpha+1)}{1-r^2}\end{aligned}$$

and

$$\begin{aligned}\min \operatorname{Re}(w) &= \operatorname{Re}\left(\frac{r^2}{1-r^2}e^{i\alpha}\right) - \frac{r}{1-r^2}\\ &= \frac{r^2\cos\alpha}{1-r^2} - \frac{r}{1-r^2} = \frac{r(r\cos\alpha - 1)}{1-r^2}.\end{aligned}$$

For $k \geqq 0$ and $\cos\alpha \geqq 0$, to get the minimum value of $\operatorname{Re}(\frac{zf'(z)}{f(z)})$, we take the minimum value of $\operatorname{Re}(w)$. Hence, we have

$$\begin{aligned}\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &= 1 + k\operatorname{Re}(w)\\ &= 1 + kr\frac{r\cos\alpha - 1}{1-r^2} > \beta.\end{aligned}$$

By a simple calculation, we have

$$(1-\beta)(1-r^2) + kr(r\cos\alpha - 1) > 0,$$

or

$$(k\cos\alpha + \beta - 1)r^2 - kr + 1 - \beta > 0. \tag{6.2}$$

If $k\cos\alpha + \beta - 1 > 0$, we get, from (6.2),

$$r < \frac{k - \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)},$$

and

$$\frac{k + \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)} < r.$$

Since

$$\begin{aligned}&\sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}\\ &= \sqrt{(k - 2(1-\beta)\cos\alpha)^2 + 4(1-\beta)^2 - 4(1-\beta)^2\cos^2\alpha}\\ &= \sqrt{(k - 2(1-\beta)\cos\alpha)^2 + 4(1-\beta)^2\sin^2\alpha} \geqq 0\end{aligned}$$

for all k and α, we consider the following inequality

$$\frac{k - \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)} > 0. \tag{6.3}$$

To be satisfied the inequality (6.3), the following inequality

$$k - \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2} > 0,$$

should be satisfied. After calculations, we have

$$4(1-\beta)(k\cos\alpha + \beta - 1) > 0.$$

The last inequality is always satisfied because $k\cos\alpha + \beta - 1 > 0$ in this case. Thus the inequality (6.3) is always satisfied. Therefore,

$$0 \leqq r < \frac{k - \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)} \qquad (k > \frac{1-\beta}{\cos\alpha}).$$

Finally, we calculate for k such that next inequality is satisfied

$$\frac{k - \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)} < 1. \tag{6.4}$$

Since the inequality (6.4) implies that

$$k - \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2} < 2(k\cos\alpha + \beta - 1),$$

we see

$$4k(1-\cos\alpha)(k\cos\alpha + \beta - 1) > 0.$$

The last inequality is always satisfied because $k\cos\alpha + \beta - 1 > 0$ in this case. Thus, the inequality (6.4) is always satisfied. Therefore, we derive

$$0 \leqq r < \frac{k - \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)} \qquad (k > \frac{1-\beta}{\cos\alpha}).$$

If $k\cos\alpha + \beta - 1 < 0$, we have from (6.2),

$$\frac{k + \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)} < r < \frac{k - \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)}.$$

Since

$$\frac{k + \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)} < 0$$

in this case, we have to have

$$\frac{k - \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)} > 0.$$

This inequality shows that

$$4(1-\beta)(k\cos\alpha + \beta - 1) < 0.$$

The last inequality is always satisfied in this case. Similary to the case $k\cos\alpha + \beta - 1 > 0$, we have to check that

$$\frac{k - \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)} < 1,$$

which implies that

$$4k(1-\cos\alpha)(k\cos\alpha+\beta-1)<0.$$

The last inequality is always satisfied in this case. Hence, we have

$$0 \leqq r < \frac{k-\sqrt{k^2-4k(1-\beta)\cos\alpha+4(1-\beta)^2}}{2(k\cos\alpha+\beta-1)} \qquad (0 \leqq k < \frac{1-\beta}{\cos\alpha}).$$

Therefore, we derive the result of the case (i) of Theorem 4 - (1).

If $k\cos\alpha+\beta-1=0$, we have, from (6.2),

$$-kr+1-\beta>0,$$

or

$$r<\frac{1-\beta}{k}=\cos\alpha.$$

We get the result of the case (ii) of Theorem 4 - (1).

If $k=0$, we have from (6.2),

$$(\beta-1)r+1-\beta>0$$

which shows $r<1$.

Therefore, we get the result of the case (iii) of Theorem 4 - (1). The proof of Theorem 4 - (1) is completed.

For $k \geqq 0$ and $\cos\alpha<0$, similary to the case (1), we derive the inequality (6.2). In this condition, $k\cos\alpha+\beta-1$ is always non-positive. Noting that

$$0 \leqq r < \frac{k-\sqrt{k^2-4k(1-\beta)\cos\alpha+4(1-\beta)^2}}{2(k\cos\alpha+\beta-1)},$$

we see

$$\frac{k-\sqrt{k^2-4k(1-\beta)\cos\alpha+4(1-\beta)^2}}{2(k\cos\alpha+\beta-1)}<1$$

if $k\neq 0$, and $0 \leqq r<1$ if $k=0$. Thus we get the result of the case (2) of Theorem 4.

For $k<0$ and $\cos\alpha \geqq 0$, to get the minimum value of $\mathrm{Re}(\frac{zf'(z)}{f(z)})$, we need the maximum value of $\mathrm{Re}(w)$. Indeed, we have

$$\begin{aligned}\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) &= 1+k\ \mathrm{Re}(w)\\ &= 1+kr\frac{r\cos\alpha+1}{1-r^2}>\beta.\end{aligned}$$

After calculations, we have

$$(1-\beta)(1-r^2)+kr(r\cos\alpha+1)>0,$$

that is,

$$(k\cos\alpha + \beta - 1)r^2 + kr + 1 - \beta > 0. \tag{6.5}$$

With this condition, similary to the case (2) of Theorem 4, $k\cos\alpha + \beta - 1$ is always positive. Solving (6.5) for r, we obtain

$$\frac{-k + \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)} < r < \frac{-k - \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)}.$$

We note that

$$\frac{-k + \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)} < 0$$

and

$$\frac{-k - \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)} > 0.$$

This gives us that

$$4(1-\beta)(k\cos\alpha + \beta - 1) < 0.$$

The last inequality is always satisfied in this condition. Finally, we calculate for k such that next inequality is satisfied

$$\frac{-k - \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)} < 1. \tag{6.6}$$

By (6.6), we have

$$-k - \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2} > 2(k\cos\alpha + \beta - 1),$$

so

$$4k(1 + \cos\alpha)(k\cos\alpha + \beta - 1) > 0.$$

The last inequality is always satisfied in this condition. Thus, the inequality (6.6) is also always satisfied. Therefore, we derive

$$\begin{aligned} 0 \leqq r &< \frac{-k - \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)} \\ &= \frac{k + \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(1 - \beta - k\cos\alpha)}, \end{aligned}$$

which is the result of the case (3) of Theorem 4.

For $k < 0$ and $\cos\alpha < 0$, we derive the inequality (6.5). If $k\cos\alpha + \beta - 1 > 0$, then we have, from (6.5),

$$r < \frac{-k - \sqrt{k^2 - 4k(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(k\cos\alpha + \beta - 1)},$$

and

$$\frac{-k+\sqrt{k^2-4k(1-\beta)\cos\alpha+4(1-\beta)^2}}{2(k\cos\alpha+\beta-1)} < r.$$

Therefore the following inequality

$$\frac{-k-\sqrt{k^2-4k(1-\beta)\cos\alpha+4(1-\beta)^2}}{2(k\cos\alpha+\beta-1)} > 0 \tag{6.7}$$

is satisfied. Since the inequality (6.7) implies

$$-k-\sqrt{k^2-4k(1-\beta)\cos\alpha+4(1-\beta)^2} > 0,$$

we have

$$4(1-\beta)(k\cos\alpha+\beta-1) > 0.$$

This last inequality is always satisfied in this case. Then, we calculate for k such that the inequality

$$\frac{-k-\sqrt{k^2-4k(1-\beta)\cos\alpha+4(1-\beta)^2}}{2(k\cos\alpha+\beta-1)} < 1$$

is satisfied. Noting that

$$-k-\sqrt{k^2-4k(1-\beta)\cos\alpha+4(1-\beta)^2} < 2(k\cos\alpha+\beta-1),$$

we have

$$4k(1+\cos\alpha)(k\cos\alpha+\beta-1) < 0.$$

The last inequality is always satisfied in this case. Hence, we have

$$\begin{aligned} 0 \leqq r &< \frac{-k-\sqrt{k^2-4k(1-\beta)\cos\alpha+4(1-\beta)^2}}{2(k\cos\alpha+\beta-1)} \\ &= \frac{k+\sqrt{k^2-4k(1-\beta)\cos\alpha+4(1-\beta)^2}}{2(1-\beta-k\cos\alpha)} \qquad (k < \frac{1-\beta}{\cos\alpha}). \end{aligned}$$

If $k\cos\alpha+\beta-1<0$, we have, from (6.5),

$$\frac{-k+\sqrt{k^2-4k(1-\beta)\cos\alpha+4(1-\beta)^2}}{2(k\cos\alpha+\beta-1)} < r < \frac{-k-\sqrt{k^2-4k(1-\beta)\cos\alpha+4(1-\beta)^2}}{2(k\cos\alpha+\beta-1)}.$$

By the same manner as in the previous cases, we have

$$\frac{-k-\sqrt{k^2-4k(1-\beta)\cos\alpha+4(1-\beta)^2}}{2(k\cos\alpha+\beta-1)} > 0,$$

which implies

$$4(1-\beta)(k\cos\alpha+\beta-1)<0.$$

The last inequality is always satisfied in this case. Thus, we have

$$\begin{aligned} 0 \leqq r &< \frac{-k-\sqrt{k^2-4k(1-\beta)\cos\alpha+4(1-\beta)^2}}{2(k\cos\alpha+\beta-1)} \\ &= \frac{k+\sqrt{k^2-4k(1-\beta)\cos\alpha+4(1-\beta)^2}}{2(1-\beta-k\cos\alpha)} < 1 \qquad \left(\frac{1-\beta}{\cos\alpha}<k<0\right). \end{aligned}$$

Therefore, we derive the result of the case (i) of Theorem 4 - (4).
If $k\cos\alpha+\beta-1=0$, we have, from (6.5),

$$kr+1-\beta>0,$$

or

$$r<-\frac{1-\beta}{k}=-\cos\alpha.$$

We get the result of the case (ii) of Theorem 4 - (4). And the proof of Theorem 4 is completed.

We give some examples for Theorem 4 as follows.

Example 5.

(1) $f(z)=\dfrac{z}{(1-z)^{5e^{i\frac{\pi}{3}}}} \in S^*(\frac{1}{7})$ for $0 \leqq r < \frac{35-\sqrt{949}}{23}=0.182355\cdots$,

(2) $f(z)=\dfrac{z}{(1-z)^{4e^{i\frac{5}{6}\pi}}} \in S^*(\frac{1}{2})$ for $0 \leqq r < \frac{-4+\sqrt{17+4\sqrt{3}}}{1+4\sqrt{3}}=0.112465\cdots$,

(3) $f(z)=\dfrac{z}{(1-z)^{-6e^{i\frac{\pi}{4}}}} \in S^*(\frac{1}{3})$ for $0 \leqq r < \frac{3\left(-6+\sqrt{\frac{340}{9}+8\sqrt{2}}\right)}{4+18\sqrt{2}}=0.102513\cdots$,

(4) $f(z)=\dfrac{z}{(1-z)^{-\frac{2}{3}e^{i\frac{2}{3}\pi}}} \in S^*(\frac{3}{7})$ for $0 \leqq r < \frac{-7+\sqrt{109}}{5}=0.688061\cdots$,

(5) $f(z)=\dfrac{z}{(1-z)^{\frac{5}{2(1+\sqrt{5})}e^{i\frac{\pi}{5}}}} \in S^*(\frac{3}{8})$ for $0 \leqq r < \cos\frac{\pi}{5}=0.809017\cdots$,

(6) $f(z)=\dfrac{z}{(1-z)^{-\frac{5}{4}e^{i\frac{2}{3}\pi}}} \in S^*(\frac{3}{8})$ for $0 \leqq r < -\cos\frac{2}{3}\pi=0.5$.

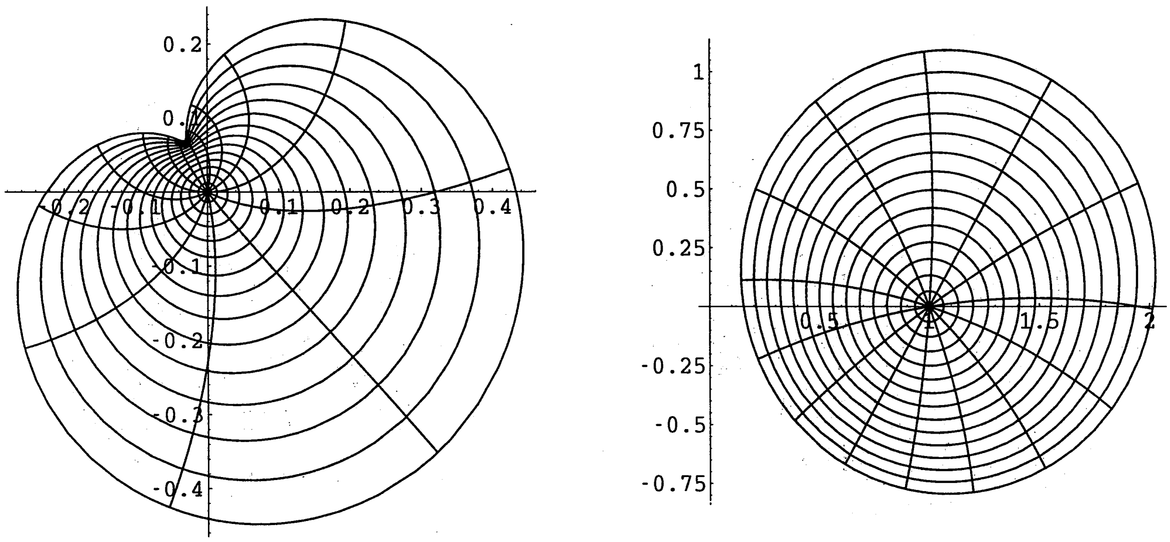

Fig 6.1: Image of $|z| = r$ by $f(z) = \dfrac{z}{(1-z)^{5e^{i\frac{\pi}{3}}}}$ (left), $\dfrac{zf'(z)}{f(z)}$ (right)

(in all cases, $r = \frac{35-\sqrt{949}}{23} = 0.182355\cdots$).

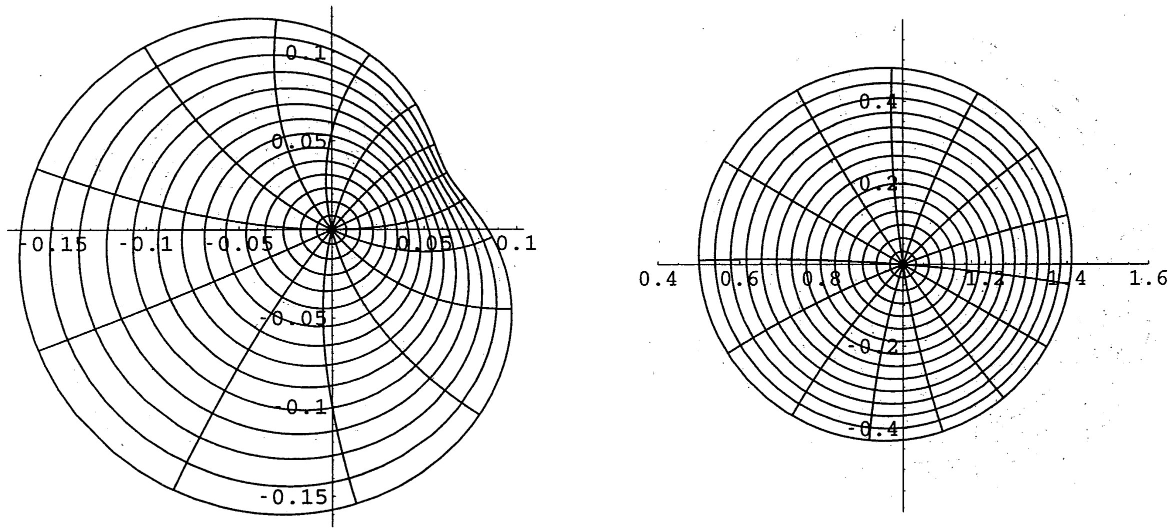

Fig 6.2: Image of $|z| = r$ by $f(z) = \dfrac{z}{(1-z)^{4e^{i\frac{5}{6}\pi}}}$ (left), $\dfrac{zf'(z)}{f(z)}$ (right)

(in all cases, $r = \frac{-4+\sqrt{17+4\sqrt{3}}}{1+4\sqrt{3}} = 0.112465\cdots$).

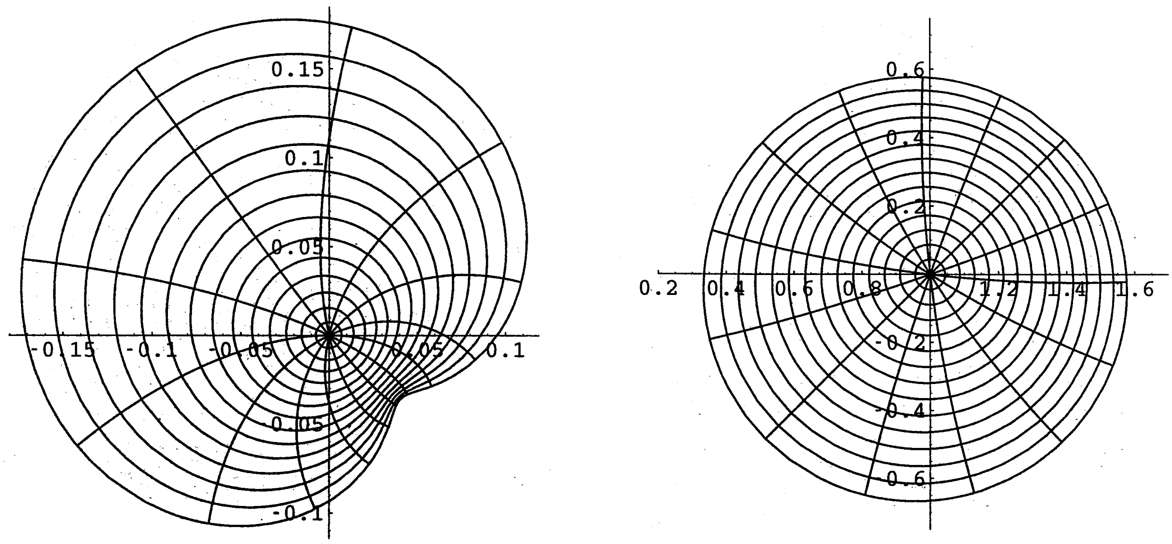

Fig 6.3: Image of $|z| = r$ by $f(z) = \dfrac{z}{(1-z)^{-6e^{i\frac{\pi}{4}}}}$ (left), $\dfrac{zf'(z)}{f(z)}$ (right)

(in all cases, $r = \frac{3\left(-6+\sqrt{\frac{340}{9}+8\sqrt{2}}\right)}{4+18\sqrt{2}} = 0.102513\cdots$).

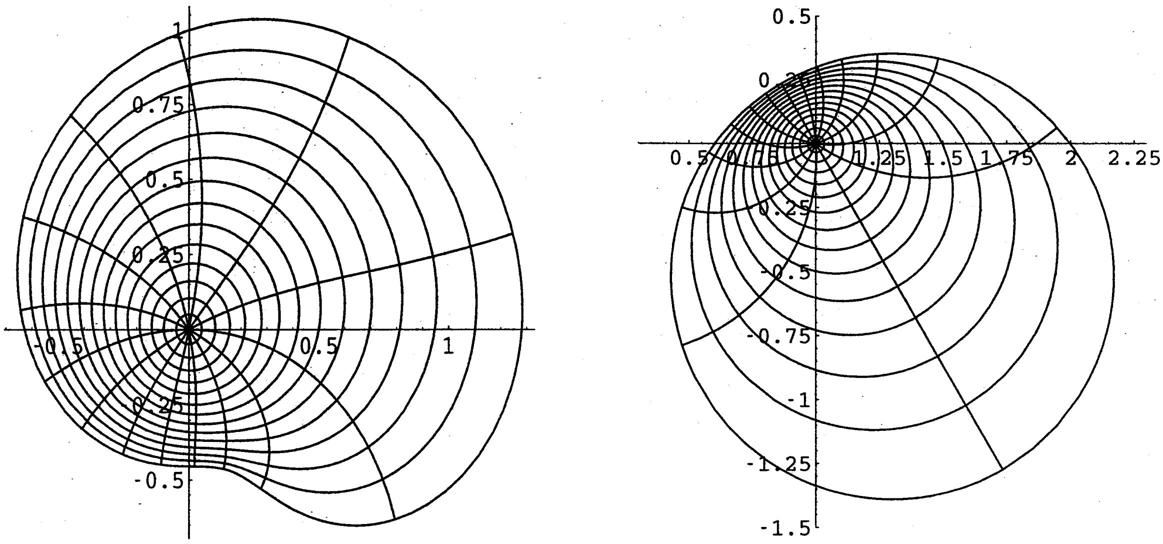

Fig 6.4: Image of $|z| = r$ by $f(z) = \dfrac{z}{(1-z)^{-\frac{2}{3}e^{i\frac{2}{3}\pi}}}$ (left), $\dfrac{zf'(z)}{f(z)}$ (right)

(in all cases, $r = \frac{-7+\sqrt{109}}{5} = 0.688061\cdots$).

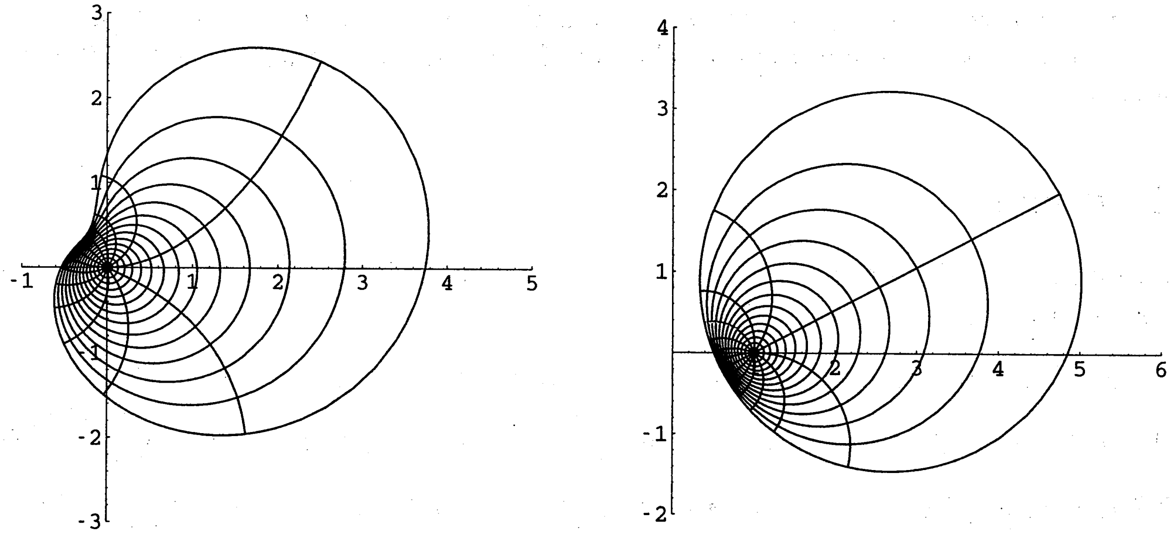

Fig 6.5: Image of $|z| = r$ by $f(z) = \dfrac{z}{(1-z)^{\frac{5}{2(1+\sqrt{5})}e^{i\frac{\pi}{5}}}}$ (left), $\dfrac{zf'(z)}{f(z)}$ (right)
(in all cases, $r = \cos\frac{\pi}{5} = 0.809017\cdots$).

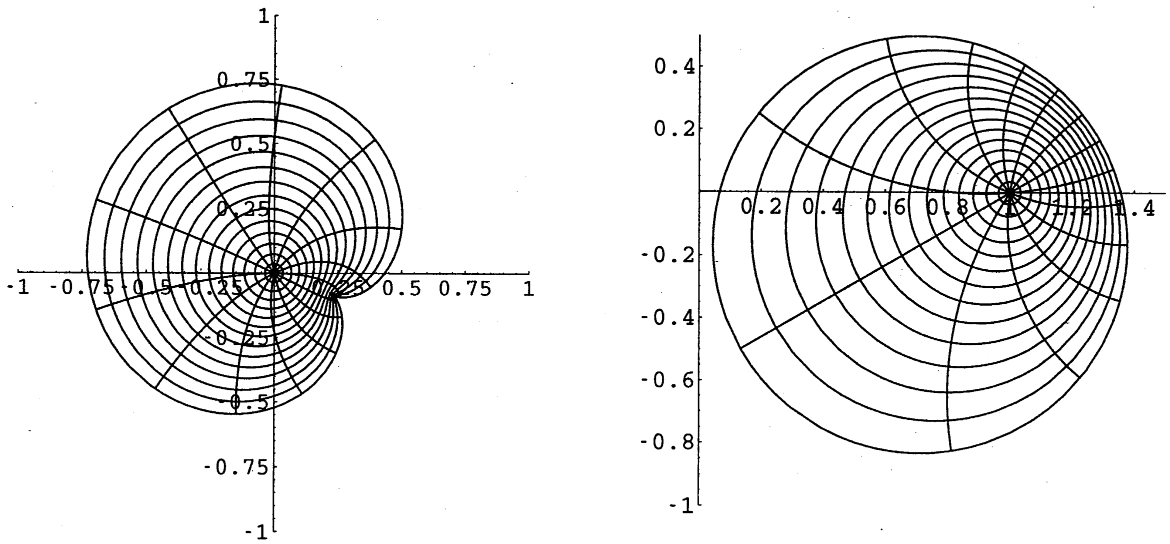

Fig 6.6: Image of $|z| = r$ by $f(z) = \dfrac{z}{(1-z)^{-\frac{5}{4}e^{i\frac{2}{3}\pi}}}$ (left), $\dfrac{zf'(z)}{f(z)}$ (right)
(in all cases, $r = -\cos\frac{2}{3}\pi = 0.5$).

Remark 1. Finally, we have to say that we can not find the sharp bound of the radius r for the classes $K_\alpha(\beta)$ and $K(\beta)$, because it is not so easy to calculate. But, as we mention before, the analytic function has the property that it maps, one-to-one, a small disk onto a small disk. Therefore, this problem to find the sharp bound of the radius r for the classes $K_\alpha(\beta)$ and $K(\beta)$ is remained.

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